BELLCOMM, INC.
1100 SEVENTEENTH STREET, N.W.   WASHINGTON, D.C.   20036

COVER SHEET FOR TECHNICAL MEMORANDUM

TITLE- Optimization of Discrete
Communication Rates

TM- 67-1014-4

DATE- August 31, 1967

FILING CASE NO(S)-   103-2

AUTHOR(S)- E. N. Shipley

FILING SUBJECT(S)-   Communication Rates
(ASSIGNED BY AUTHOR(S)-

ABSTRACT

A communication system in which the modulation band-width is one of n constant rates is considered. The usable information transfer rate is bounded by an inverse square dependence on time. Under this circumstance, the n rates which provide maximum information transfer are determined for two cases: 1) the n rates are integral multiples of a fundamental rate, the highest rate being n times the fundamental rate, and 2) the case where the n rates are independent.
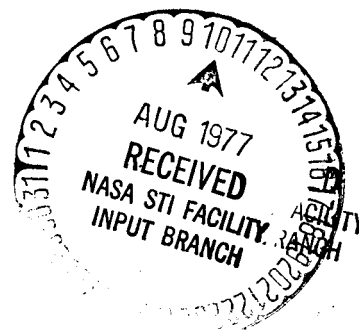
SEE REVERSE SIDE FOR DISTRIBUTION LIST

BELLCOMM, INC.

DISTRIBUTION

COMPLETE MEMORANDUM TO

CORRESPONDENCE FILES:

OFFICIAL FILE COPY
plus one white copy for each
additional case referenced

TECHNICAL LIBRARY (4)

NASA Headquarters
Messrs. A. J. Calio/SY
P. E. Culbertson/MLA
J. H. Disher/MLD
F. P. Dixon/MTY
P. Grosz/MTL
E. W. Hall/MTS
T. A. Keegan/MA-2
D. R. Lord/MTD
M. W. Molloy/SY
O. W. Nicks/SY
M. J. Raffensperger/MTE
L. Reiffel/MA-6
A. D. Schnyer/MTV
G. S. Trimble/MT

Manned Spacecraft Center
M. A. Silveira/ET25
W. E. Stoney, Jr./ET
J. M. West/AD

Marshall Space Flight Center
H. G. Hamby/R-AS-VL
R. J. Harris/R-AS-VP
B. G. Noblitt/R-AERO-XA
F. L. Williams/R-AS-DIR

Kennedy Space Center
J. P. Claybourne/EDV4
R. C. Hock/PPR2
N. P. Salvail/MC

COMPLETE MEMORANDUM TO

Electronics Research Center
R. C. Duncan/G

Langley Research Center
W. R. Hook/60.300

Ames Research Center
L. Roberts/M   (2)

Bellcomm, Inc.
F. G. Allen
G. M. Anderson
A. P. Boysen, Jr.
R. K. Chen
J. P. Downs
D. R. Hagner
P. L. Havenstein
W. G. Heffron
H. A. Helm
J. J. Hibbert
W. C. Hittinger
B. T. Howard
D. B. James
K. E. Martersteck
R. K. McFarland
J. Z. Menard
I. D. Nehama
G. T. Orrok
I. M. Ross
R. L. Selden
R. V. Sperry
J. M. Tschirgi
W. B. Thompson
R. L. Wagner
J. E. Waldo
All members, Division 101
Department 1023
Library
Central Files

SUBJECT: Optimization of Discrete
Communication Rates
Case 103-2

DATE: August 31, 1967

FROM: E. N. Shipley
TM-67-1014-4

TECHNICAL MEMORANDUM

I. INTRODUCTION

One of the problems which arises in the design of
space missions is the choice of communication bandwidth. In
general the choice will be a compromise between the desire to
transmit large quantities of data and the requirement to
restrict weight and power consumption.

In the case of many missions, and for the ones with
which we will be concerned here, the distance between the
transmitter and the receiver does not remain constant. Once
the transmitter and receiver, the antenna, the modulation
techniques, and numerous other items have been selected, the
available communication rate at a given signal to noise ratio
depends, at least, on the distance.

For several cases of interest in deep space flight
the velocity between the transmitter and receiver is constant
with time and directed along their line of centers. In par-
ticular, this is the case during a manned planetary flyby
mission when the manned vehicle communicates with an unmanned
probe at the planet subsequent to periapsis passage. As the
separation distance increases linearly with time, received
power will vary inversely with the square of time. If, in
addition, the noise background remains constant (a non-trivial
assumption), and all other transmission parameters remain con-
stant, then the rate at which information can be transmitted,
at a specified error rate, will obey the relation

$$y = at^{-2}$$

(1)

where y is the rate of information transfer, and a is a constant
which depends on the parameters of the transmission system, in-
cluding the allowed error rate. The situation is illustrated

in Figure 1. Transmission rates which at a given time exceed
y would produce excessive error rates and are therefore not
allowed; any lower rate is permitted. Thus we will speak of
equation (1) as providing the maximum permissible rate of in-
formation transfer.

The developments of the subsequent sections apply to
any situation in which the available communication bandwidth
varies with time in the fashion of equation (1).

Suppose transmissions begin at T_O greater than zero.
(The time T_O will play an essential role in the subsequent
development.) The maximum amount of information, Q, which can
be transmitted without violating (1) is given by

$$Q = \int_{T_O}^{\infty} y \, dt$$

$$Q = \frac{a}{T_O} \tag{2}$$

Q is the total information which can be transmitted, beginning
at T_O and continuing indefinitely, under the prescribed condi-
tions. It requires the information rate which is transmitted
be continuously adjusted to the rate indicated by equation (1).
Q will serve as a benchmark against which other operational
schemes can be compared.

Our concern is with modulation systems for which it
is not possible to adjust the information rate continuously.
Instead, it is assumed that the system is capable of transmitting
at any one of n discrete rates; these rates are constant and are
determined prior to the mission, but may be freely chosen then.
The question which is posed is: how should the n rates be chosen
in order to maximize the transmission of information under the
restriction of equation (1). It is an inherent assumption in
this discussion that it is desirable to maximize the flow of
information, and that data transmitted at any time is as useful
as data transmitted at another time.

Two separate cases are considered in subsequent sections. In the first, the n rates are restricted to be integral multiples of a fundamental rate; the highest rate is n times the fundamental rate. This is the "Integral Multiple Case". In the second "Arbitrary Rate Case", the n transmission levels may be chosen independently of one another.

This entire problem has been suggested by work done on the communication aspects of Mars flyby missions.[1,2] The communication rates used in the "Interim Mission Sequence Plan"[1] were chosen prior to the present study and were not optimized in the sense used here. However, their choice was based on many careful considerations, including hardware limitations and data requirements. In the present study, the sole consideration has been the maximization of the data transmission. As such, it should be considered only as a component part of an overall optimization of the transmission.

II. INTEGRAL MULTIPLE LEVELS

In this section, we make the assumption that the transmission may take place at any of n levels which are integral multiples of a fundamental rate $u^{(n)}$. That is, the n rates are given by

$$u^{(n)}, \; 2u^{(n)}, \; 3u^{(n)}, \; . \; . \; . \; . \; . \; nu^{(n)} \qquad (3)$$

We wish to maximize the total information flow $Q^{(n)}$ by appropriately choosing the fundamental rate u.

Such a system as described above has been proposed for relaying data between a Mars photographic orbiting vehicle and the flyby vehicle. In that case, the fundamental rate corresponds to the rate at which a single scanning device in the Mars orbiter converts photographic information to analog data. It is supposed that this device works at a constant rate, although in our case we suppose that the rate may be freely chosen before the start of the mission. Because of hardware constraints such a circumstance may not exist in fact.

   In order to increase the data flow from the orbiter
to the flyby vehicle, several scanning devices working in
parallel may be employed. The number of devices in operation
at any one time may be changed; thus, if there are n scanning
devices, then transmission may occur at any one of the n rates
given in equation (3).

   We label the rates by the relation

$$\rho_j{}^{(n)} = (n + 1 - j)\, u^{(n)} \tag{4}$$

where $u^{(n)}$ is the fundamental rate in the case of n levels. We
shall commonly suppress the superscript (n) where it does not
add to the clarity of exposition. The fundamental rate which
maximizes the information transfer will be denoted by $r^{(n)}$.

   Each rate intercepts the curve of permissible rates
(equation 1) at a time T_j (see Figure 2) given by

$$\rho_j = aT_j{}^{-2} \tag{5}$$

and we have immediately

$$T_j = [a/(n+1-j)u]^{1/2} \cdot \qquad j = 1,\ldots,n \tag{6}$$

and we note that $T_{j+1} > T_j$.

   Since it is desired to maximize the total information
flow, we can consider that the rate ρ_j is utilized during the
period

$$T_{j-1} < t \le T_j$$

A lower rate would provide less information, and the next higher rate intercepts the permissible limit curve (equation (1)) at T_{j-1}, and so is not permitted in the time interval.

The area q under the step function curve of Figure 2 is equal to the total information transmitted, and the area is given by

$$q^{(n)} = \sum_{j=1}^{n} (\rho_j - \rho_{j+1})(T_j - T_o) \tag{7}$$

We seek to maximize $q^{(n)}$ by an appropriate choice of the ρ_j, that is, through equation (4), by choice of the fundamental rate $u^{(n)}$. For this purpose, we will restrict u to the interval

$$0 < u^{(n)} < \frac{a}{nT_o^2} \tag{8}$$

since this guarantees that $T_1 \geq T_o$, and because $T_{j+1} \geq T_j$, that all $T_j \geq T_o$. Without the limitation (8), equation (7) contains undesirable terms which correspond to negative information transfer. At a later point we will carefully consider the effect of the restriction (8).

Consider equation (7); the term $(\rho_j - \rho_{j+1})$ is just equal to u, and we obtain, with the use of (6),

$$q(u) = u \sum_{j=1}^{n} \left\{ [a/(n + 1 - j)u]^{1/2} - T_o \right\} \tag{9}$$

To maximize the area, we set

$$\frac{dq}{du} = 0$$

This is a necessary but not sufficient condition for the existence of an extremum. We shall later show that in fact a maximum is obtained, but in the meantime we will assume that it provides a maximum point. From the previous equation one obtains

$$r^{(n)} = \frac{a}{4T_o^2 n^2} \left[\sum_{j=1}^{n} \frac{1}{(n + 1 - j)^{1/2}} \right]^2 \tag{10}$$

where $r^{(n)}$ is the value of u which maximizes $q(u)$. We note that the sum in equation (10) may be written in the form

$$\sum_{j=1}^{n} \frac{1}{(n + 1 - j)^{1/2}} = \sum_{k=1}^{n} \frac{1}{k^{1/2}}$$

We then define

$$S_n = \frac{1}{n} \left[\sum_{k=1}^{n} \frac{1}{k^{1/2}} \right]^2 \tag{11}$$

so that

$$r^{(n)} = \frac{a}{4nT_o^2} S_n \tag{12}$$

The total amount of information which is transmitted when the optimum fundamental rate is employed can be obtained from equation (9)

$$q_m^{(n)} = (ar)^{1/2} \left(\sum_{j=1}^{n} \frac{1}{(n + 1 - j)^{1/2}} \right) - nT_o r$$

where we have defined

$$q_m^{(n)} = q(u = r^{(n)})$$

Using 12, we find

$$q_m^{(n)} = \frac{a}{4T_o} S_n \tag{13}$$

The function S_n has been evaluated using the Bellcomm 7040-7044 digital computer. Double precision arithmetic was used to insure the accuracy of the results, which are given in Table I for certain values of n, and which are plotted in Figure 3.

It is interesting to consider the limit of the optimum process as $n \rightarrow \infty$. It is shown in Appendix A that

$$\underset{n \rightarrow \infty}{\text{Lim}} S_n = 4 \tag{14}$$

Not unexpectedly, we find then

$$\underset{n \rightarrow \infty}{\text{Lim}} q_m^{(n)} = \frac{a}{T_o} = Q$$

which is just the area under the maximum permissible curve (equation 1) from T_o to ∞. We also see

$$\underset{n \rightarrow \infty}{\text{Lim}} r^{(n)} = 0$$

and

$$\lim_{n \to \infty} \rho_j = \lim_{n \to \infty} (n+1-j)r^{(n)} = \frac{a}{T_j^2}$$

The latter result merely states that in the limit of large n, the transmission rate at any specific time T_j is given by the rate of the maximum permissible curve at T_j.

Suppose the fundamental rate differs from the value which provides a maximum information transfer. Let us define the efficiency E to be

$$E^{(n)} \left(\frac{u}{r} \right) = q^{(n)}(u)/q_m^{(n)} \tag{15}$$

With the use of equations (9), (11), (12) and (13) we find

$$E \left(\frac{u}{r} \right) = 2 \left(\frac{u}{r} \right)^{1/2} - \left(\frac{u}{r} \right) \tag{16}$$

This relationship is plotted in Figure 4. This equation shows that the point r = u, where $\frac{dq}{du} = 0$, is in fact a real maximum. We had made that assumption earlier in this section.

We recall that equation (8) limits the range of u and equation (16) is valid only for

$$0 \leq u \leq \frac{a}{nT_o^2} \tag{8}$$

However, we are now in a position to discuss this restriction. The upper limit of the fundamental rate in equation (8) is chosen so that the rate ρ_1 satisfies the inequality

$$\rho_1 = nu \leq \frac{a}{T_o^2}$$

Now, if u is allowed to increase and to lie in the range

$$\frac{a}{nT_o^2} < u^{(n)} \leq \frac{a}{(n-1)T_o^2} \cdot \qquad (17)$$

then we have

$$\rho_1 > \frac{a}{T_o^2} \quad ,$$

$$\rho_2 \leq \frac{a}{T_o^2}$$

That is, there are exactly (n-1) transmission levels which are useful for times greater than T_o. The nth level, ρ_1, exceeds the maximum permissible rate except for $t \leq T_o$, and because we assume no transmissions before T_o, is excluded from use. Thus in the interval given by (17), for the n-level case the amount of information transmitted is the same as for the n-1 level case. We are faced with the following question: in the case of n levels, is it possible to transmit more information by a choice of u which restricts the number of usable levels to n-1?

The answer is no. It is shown in Appendix A that

$$S_n > S_{n-1} \qquad (18)$$

Consequently, since

$$q_m{}^{(n)} = \frac{a}{4T_o} \; S_n$$

we have

$$q_m{}^{(n)} > q_m{}^{(n-1)} \tag{19}$$

where for $q_m{}^{(n)}$ the restriction on u is given by (8) and for $q_m{}^{(n-1)}$ by

$$0 \le u < \frac{a}{(n-1)T_o{}^2} \tag{20}$$

For the n-level case, however, if u lies in the range of equation (17), the information transmitted must be less than or equal to $q_m{}^{(n-1)}$ and, through equation (19), less than $q_m{}^{(n)}$. Thus we are now insured that the fundamental rate $r^{(n)}$ provides a maximum information transfer for any u in the interval given by equation (20).

It is possible to extend the range of u by subsequent arguments for n-2 levels, then n-3 and so forth, until we reach the point where u exceeds $a/T_o{}^2$, at which point no transmission at all is possible.

That is, we have verified that $r^{(n)}$ does indeed produce a maximum amount of information transmittal in the case where at most n-levels are allowed, and where there is no restriction on the range of u. Equation (16), however, is only applicable to the range given by (8).

III.   ARBITRARY RATES

The problem which will be discussed in this section is the same as before, except that we release the constraint that the transmission rates be integral multiples of a fundamental rate. We consider that there are n transmission rates σ_j, $j = 1, \cdots, n$, satisfying

$$\sigma_j > \sigma_{j-1}$$

It is also convenient to define

$$\sigma_o = aT_o^{-2} \tag{21a}$$

and

$$\sigma_{n+1} = 0 \tag{21b}$$

We also define T_j as the time when rate σ_j intersects the curve giving the maximum permissible rate (equation 1), i.e.,

$$\sigma_j = aT_j^{-2} \tag{21c}$$

The rate σ_j is used during the period T_{j-1} to T_j. This provides the maximum information transfer in the interval since the next higher rate exceeds the maximum permissible rate in the interval. Note that equations (21a) and (21c) are consistent.

The total information which is transmitted is

$$w^{(n)} = \sum_{j=1}^{n} \sigma_j (T_j - T_{j-1}) \qquad (22)$$

and we seek that set of n σ_j's for which $w^{(n)}$ is a maximum. Since $w^{(n)}$ has continuous first derivatives, the maximum point must solve the equation

$$\frac{\partial w^{(n)}}{\partial \sigma_j} = 0 \qquad (23)$$

We obtain n equations of the form

$$\frac{1}{2}\left(\frac{a}{\sigma_j}\right)^{1/2} - \left(\frac{a}{\sigma_{j-1}}\right)^{1/2} + \frac{1}{2}\frac{\sigma_{j+1}}{a}\left(\frac{a}{\sigma_j}\right)^{3/2} = 0 \quad j=1, \cdot \cdot \cdot, n \qquad (24)$$

where use has been made of the definitions contained in equations (21a) and (21b), and equation (21c) has been used to eliminate T_j.

Equation (23) is a sufficient but not necessary condition to be satisfied by a maximum point; a saddle point would also satisfy the equation.[3] However, $w^{(n)}$ in equation (22) must have a maximum, because it is bounded by the area under the maximum permissible rate curve (equation 2) and it can be shown that $w^{(n)}$ approaches zero as all the $\sigma_j \to 0$ or as all the $\sigma_j \to \sigma_o$. Moreover, the maximum cannot lie on any boundary of the range of the σ's, since straightforward arguments show that small adjustments away from the boundary increase the value of $w^{(n)}$. Hence, there must be a maximum point satisfying (23), and since, as will be seen, a unique point is developed from equation (24), that must be the maximum point.

Suppose we relate σ_j to σ_{j+1} by the relation

$$\sigma_j = \Gamma_{n-j}\sigma_{j+1} \qquad j = 0, \cdot \cdot \cdot, n-1 \qquad (25)$$

By setting $j = n$ in equation (24) and making use of (25) we obtain

$$\Gamma_1 = 4 \qquad (26)$$

For arbitrary j we obtain the recursion relation

$$\Gamma_{j+1} = \left(\frac{2\Gamma_j}{1 + \Gamma_j}\right)^2 \qquad (27)$$

We can extend equation (25) as follows

$$\sigma_j = \Gamma_{n-j}\Gamma_{n-j-1}\Gamma_{n-j-2} \cdot \cdot \cdot \Gamma_1 \sigma_n$$

or

$$\sigma_j = \alpha(n - j)\sigma_n \qquad j = 0, \cdot \cdot \cdot, n \qquad (28)$$

where

$$\alpha(k) = \Gamma_k \Gamma_{k-1} \cdot \cdot \cdot \Gamma_1 \qquad k \geq 1 \qquad (29)$$

and

$$\alpha(0) = 1$$

We also have the following relation

$$\sigma_o = \alpha(n)\sigma_n$$

so that we can obtain σ_j in terms of σ_o, which is known at the outset of the problem (equation 21a).

$$\sigma_j = \frac{\alpha(n-j)}{\alpha(n)} \ \sigma_o \tag{30}$$

It should be noted at this point that the quantities Γ_k and $\alpha(k)$ are independent of n, the number of levels in the system under consideration. This simplifies the problem of tabulation of the quantities and it allows the development of these quantities without considering a specific problem.

The σ_j's of equation (30) maximize the total information transmitted, equation (22). If we write the maximum area as $W_m^{(n)}$, we can define $P^{(n)}$ as the fraction of the maximum possible information Q (equation 2) through

$$W_m^{(n)} = P_n Q = P_n \frac{a}{nT_o} \tag{31}$$

and with the use of relations (29) and (30) we find

$$P_n = \frac{1}{[\alpha(n)]^{1/2}} \sum_{m=1}^{n} \left[\alpha(m-1)\right]^{1/2} \left[1 - \Gamma_m^{-1/2}\right] \tag{32}$$

Let us calculate $\Gamma_n P_n$ from (32), and express it in terms of Γ_{n-1} and P_{n-1}. We obtain

$$\Gamma_n P_n = 1 + (2\Gamma_{n-1}P_{n-1} - 2)$$

That is, if $\Gamma_{n-1}P_{n-1} = 1$, then also $\Gamma_n P_n = 1$. But $\Gamma_1 P_1 = 1$, so the relation can be extended to all n, and we have

$$P_j = \frac{1}{\Gamma_j} \tag{33}$$

The value of the quantities Γ_j, $\alpha(j)$ and P_j are given in Table II for certain values in the range $j = 1$ to $j = 100,000$. The quantity P_n is plotted in Figure 5 with a comparison from the "integral multiple" case.

It is shown in Appendix B that $\Gamma_{n+1} < \Gamma_n$, as expected; in consequence we have

$$P_{n+1} > P_n \tag{34}$$

This means that to increase the total information transmission, it is advantageous to increase the number of levels at which transmissions occur.

The limit of P_n as $n \to \infty$ can be determined as follows. Certainly, because of its definition, $P_n \leq 1$, because the transmission rates are always less than the maximum permissible value (equation 1). On the other hand,

$$w_m^{(n)} \geq q_m^{(n)}$$

because the levels for w_m are chosen to maximize the area; one possible set of levels is just that which maximizes the "integral multiple" case. Consequently, from equations (13), (14), and (31), we have

$$P_n \frac{a}{T_o} \geq \frac{S_n}{4} \cdot \frac{a}{T_o}$$

$$1 \geq \lim_{n \to \infty} P_n \geq \lim_{n \to \infty} \frac{S_n}{4} = 1$$
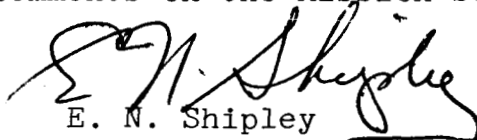
So,

$$\lim_{n \to \infty} P_n = 1$$

IV. CONCLUSION

The optimized transmission process which has been developed here was suggested by the communication procedures adopted for the Mars Flyby Mission Sequence Plan (Reference 1). In the flyby mission, the Mission Module communicates with probes which remain in the vicinity of Mars, and the available communication bandwidth decreases as the Mission Module and probes separate at essentially a constant velocity. The two modes of transmission, the integral multiple case and the arbitrary case, were used in the Mars flyby example for communication between the photographic orbiter and the Mission Module, the former corresponding to several mechanical photographic scanners operating in parallel, and the latter to readout from a buffer storage at several different rates.

A primary assumption of the scheme is that the quantity to be optimized is the total amount of information. This is a restrictive assumption and is clearly not valid in many circumstances. For example, a Mars probe may be measuring the local temperature. It may then be desired to obtain the measurements over as long a time span as possible, rather than obtain the greatest number of measurements (the greatest information).

In general, the situation for which the optimization scheme is appropriate probably involves storage of information in some fashion. That is, there is a large amount of data which is collected and stored very quickly, and the concern is transmitting as much of the stored information as possible. A photographic orbiter is, of course, a straightforward example of such a system.

ACKNOWLEDGEMENTS

The author is indebted to B. G. Smith for a proof contained in Appendix A, to C. J. Byrne for discussions on the effect of modulation techniques on signal to noise ratios, and to E. M. Grenning for comments on the Mission Sequence Plan.

1014-ENS-amb   E. N. Shipley

Attachments
    Tables I - II
    Figures 1 - 5
    Figure B-1

BELLCOMM, INC.

REFERENCES

1. E. M. Grenning, <u>Interim Mission Sequence Plan for the Encounter Portion of the 1975 Manned Mars Flyby Mission</u>, Bellcomm Memorandum for File, June 15, 1967.

2. R. K. Chen and R. L. Selden, <u>Communication Systems Design for Manned Mars Flyby Mission</u>, Bellcomm Technical Memorandum, TM-66-2021-8, July 29, 1966.

3. See, for example, R. Creighton Buck, <u>Advanced Calculus</u>, (McGraw-Hill Book Company, 1965), 2nd Edition, p. 349 ff.

TABLE I

n	S_n	n	S_n
1	1.00000000	95	3.44261798
2	1.45710677	100	3.45573369
3	1.73958133	150	3.54981032
4	1.93830025	200	3.60709852
5	2.08873901	250	3.64671883
6	2.20816830	300	3.67624095
7	2.30619815	400	3.71807867
8	2.38868245	500	3.74689883
9	2.45942909	600	3.76831371
10	2.52104199	700	3.78504023
12	2.62378249	800	3.79857606
14	2.70670149	900	3.80982390
16	2.77555150	1000	3.81936467
18	2.83397138	1500	3.85190520
20	2.88439494	2000	3.87143192
22	2.92852223	2500	3.88481301
24	2.96758291	3000	3.89471945
26	3.00249204	4000	3.90866646
28	3.03394714	5000	3.91821226
30	3.06249067	6000	3.92527324
32	3.08855200	7000	3.93076956
34	3.11247611	8000	3.93520549
36	3.13454390	9000	3.93888351
38	3.15498680	10000	3.94199762
40	3.17399749	15000	3.95257971
45	3.21626109	20000	3.95890105
50	3.25246081	25000	3.96322054
55	3.28393462	30000	3.96641201
60	3.31163856	40000	3.97089604
65	3.33627671	50000	3.97395888
70	3.35838097	60000	3.97622126
75	3.37836179	70000	3.97798046
80	3.39654168	80000	3.97939906
85	3.41317827	90000	3.98057446
90	3.42848006	100000	3.98156908

VALUE OF THE SEQUENCE S_n FOR SELECTED n. THE SEQUENCE
IS DISCUSSED IN THE TEXT.

NOT ALLOWED REGION

$y = at^{-2}$

ALLOWED REGION

Transmission Rate (Arbitrary Units)
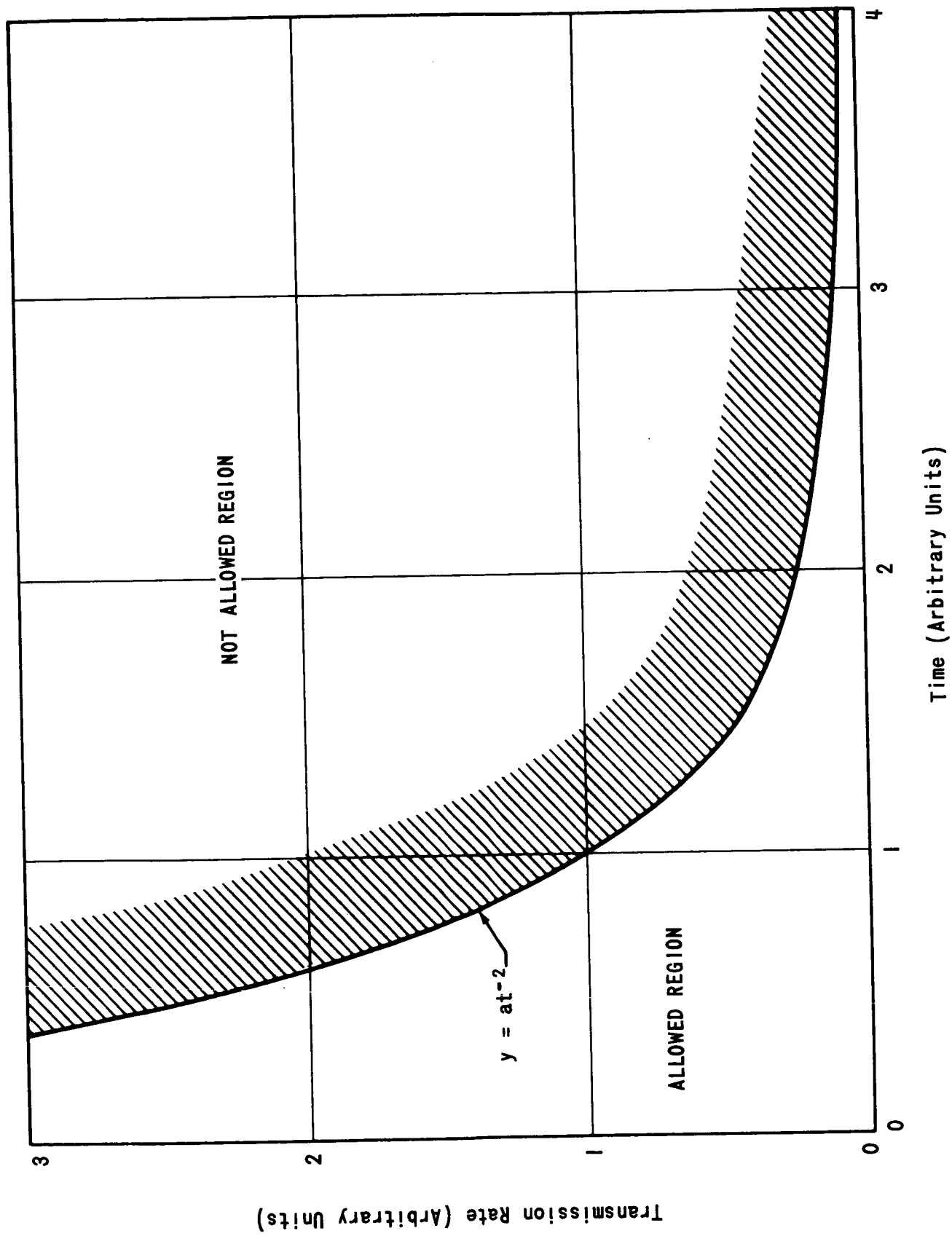
Time (Arbitrary Units)

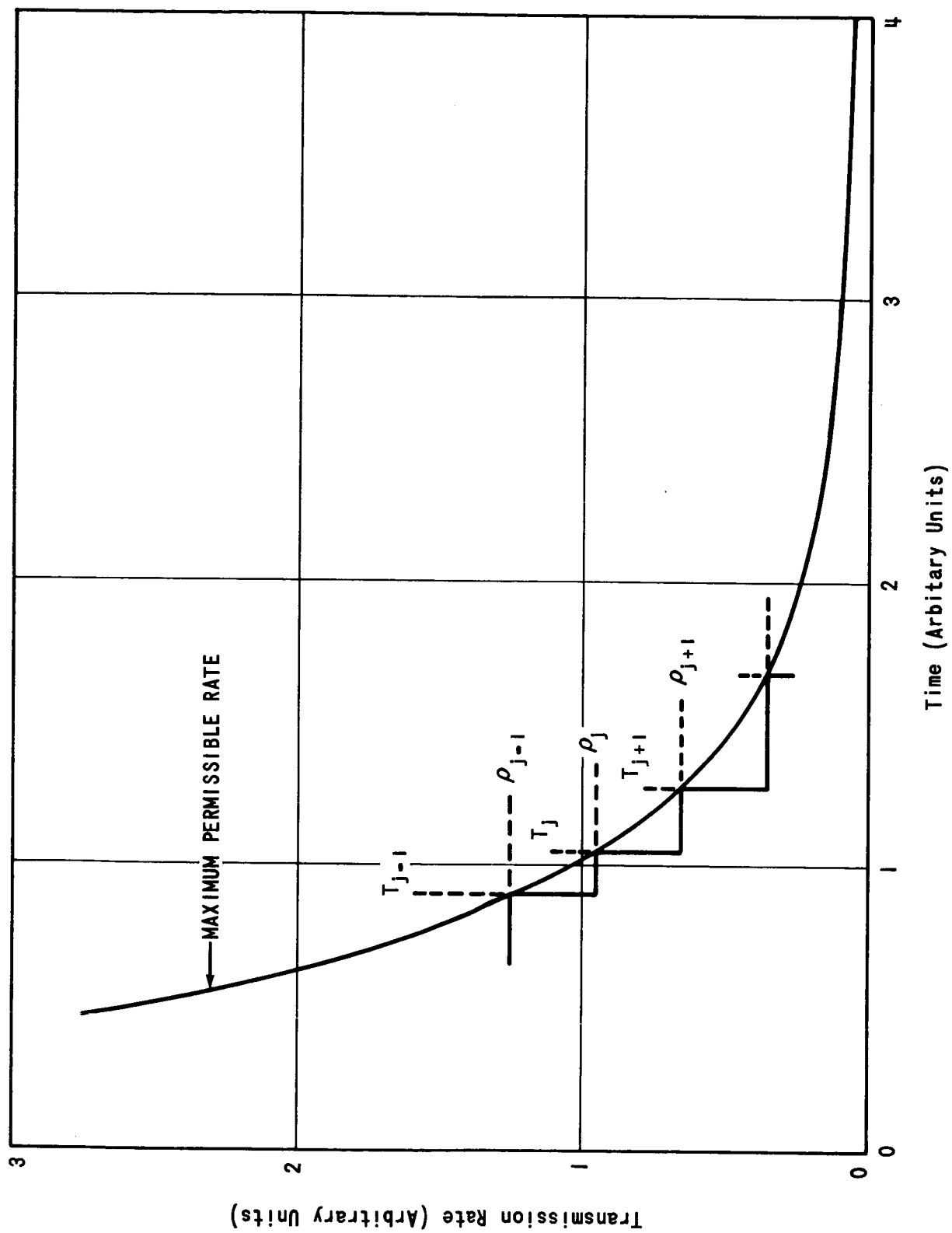FIGURE 1   ILLUSTRATION OF ALLOWED AND UNAVAILABLE REGIONS OF TRANSMISSION RATE

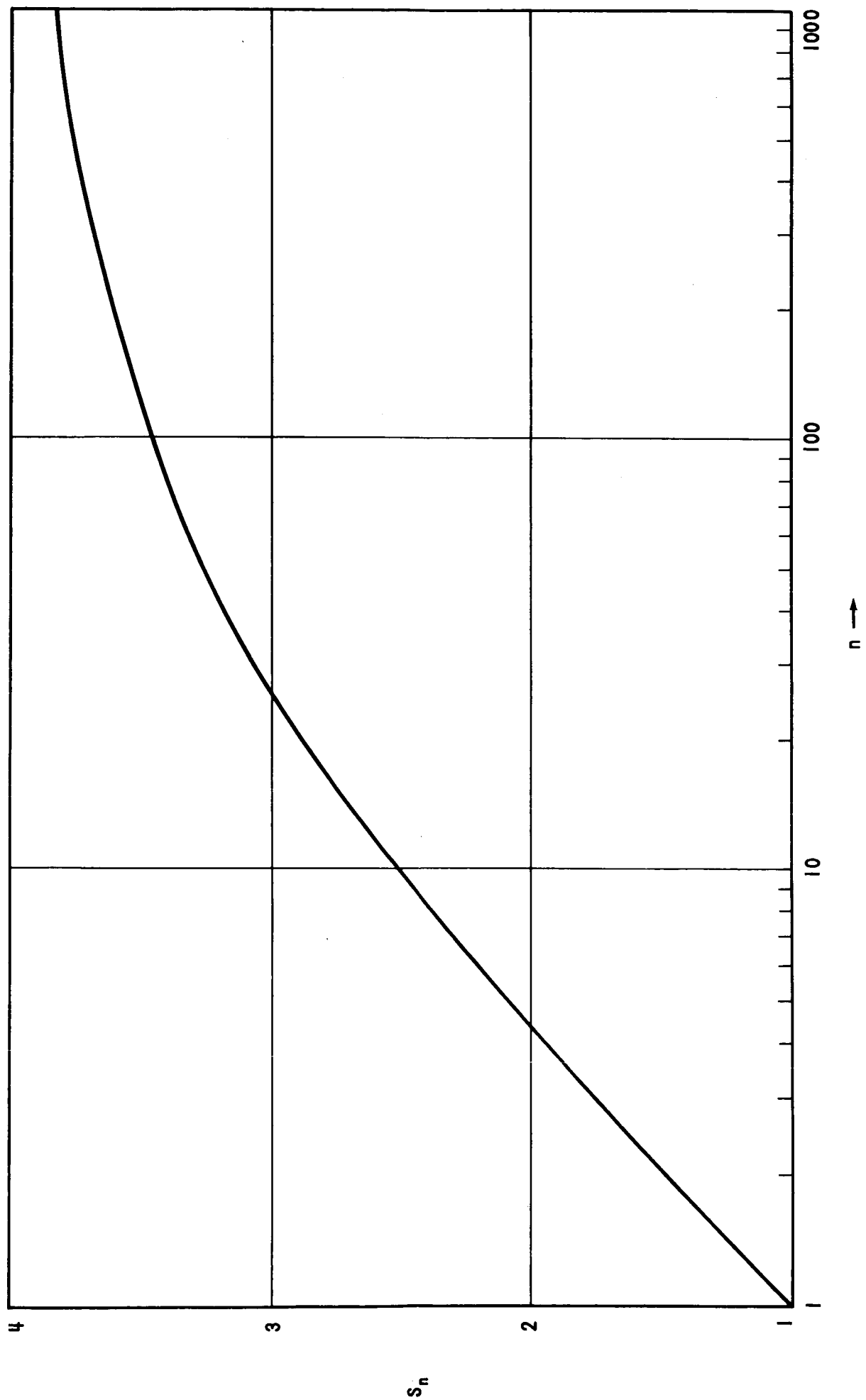FIGURE 2 RELATIONSHIP OF THE RATES ρ_j AND THE TIMES T_j

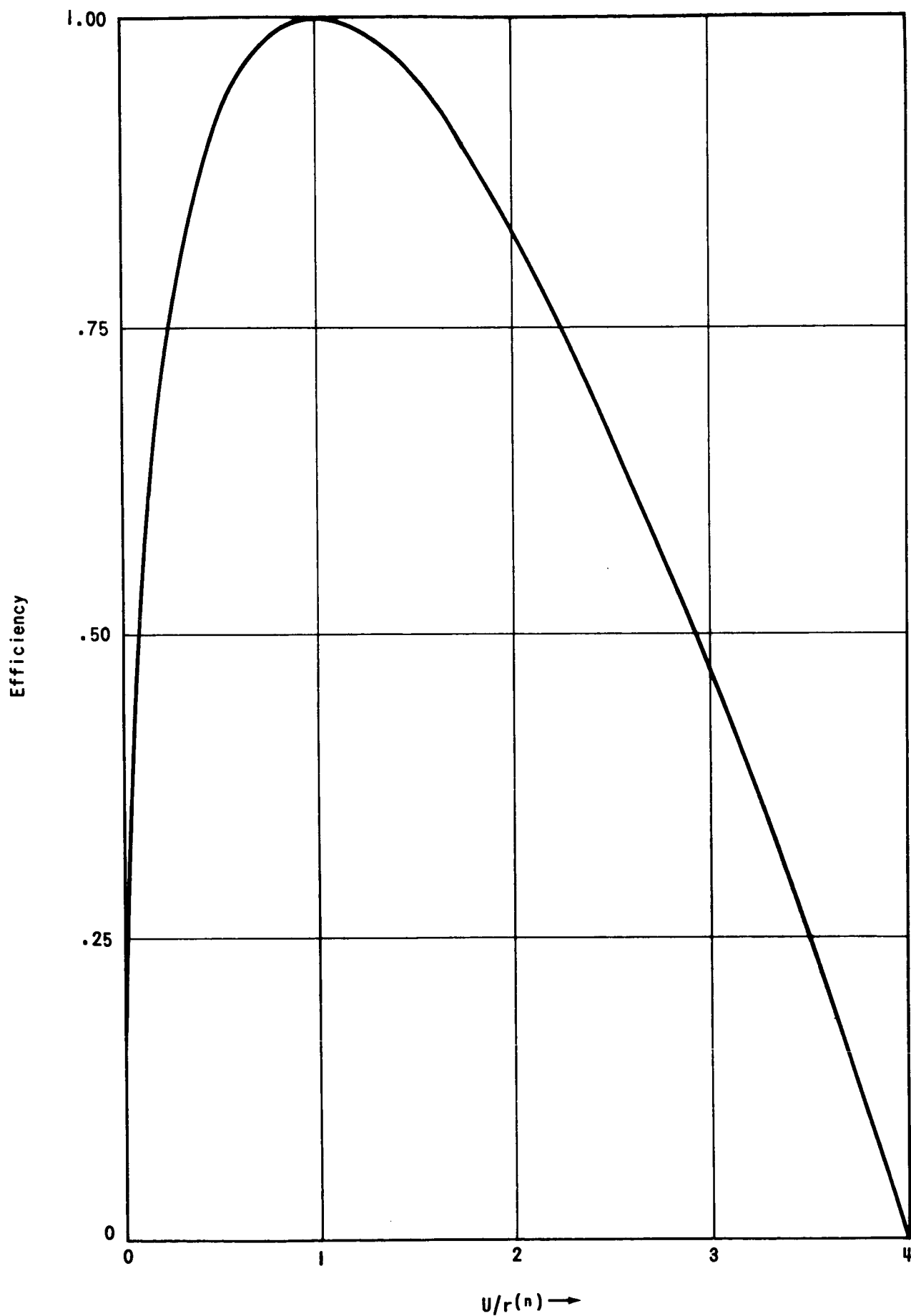FIGURE 3   THE SEQUENCE S_n PLOTTED AS A FUNCTION OF n

FIGURE 4   EFFICIENCY PLOTTED AS A FUNCTION OF THE RATIO OF ACTUAL
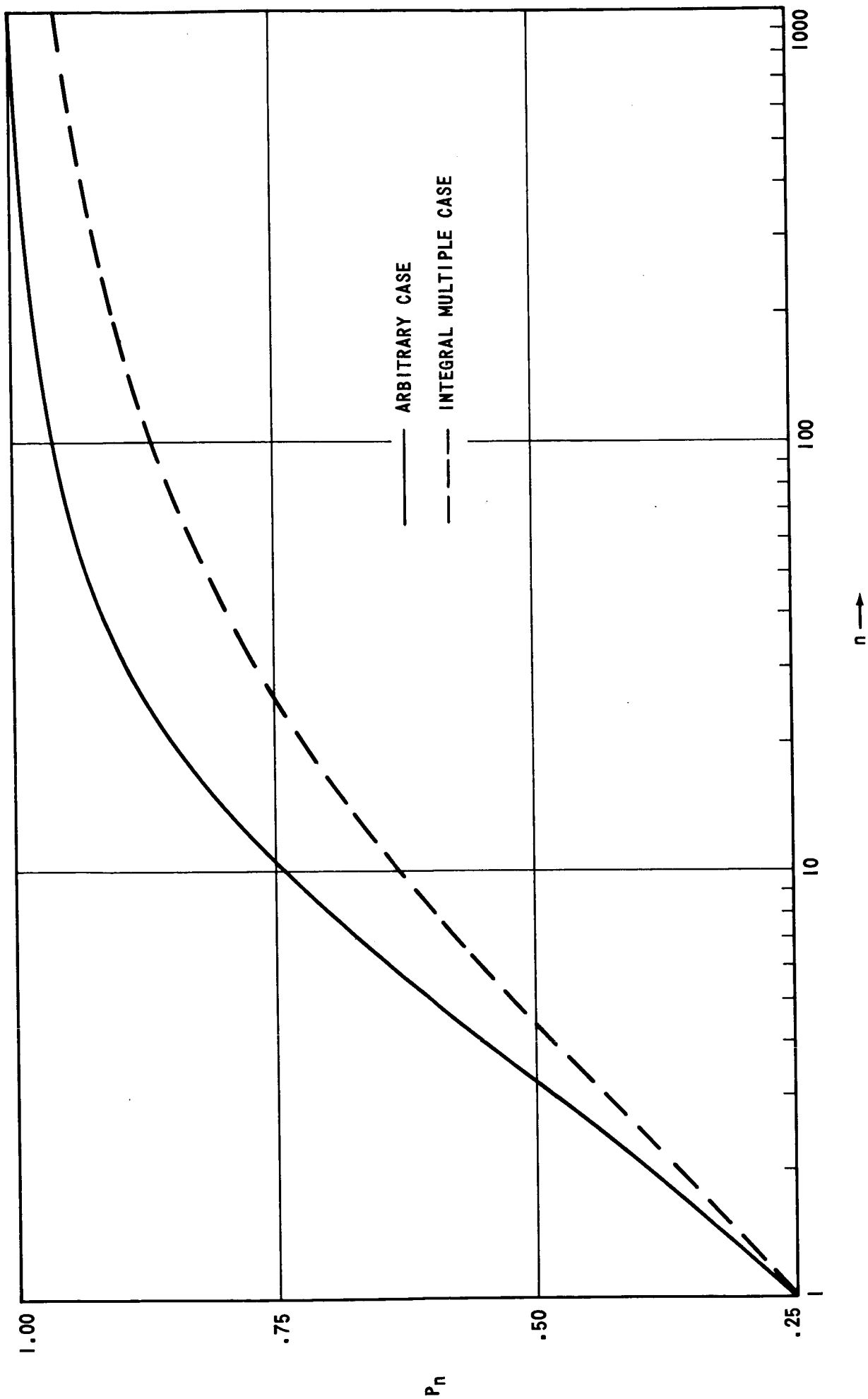TO OPTIMUM FUNDAMENTAL RATE. THE CURVE IS VALID FOR
$U \leq a/nT_0^2$

FIGURE 5   THE FRACTION, P_n, OF THE MAXIMUM PERMISSIBLE AREA WHICH IS
TRANSMITTED IN THE ARBITRARY AND IN THE INTEGRAL MULTIPLE CASE

TABLE II

j	P_j	$\alpha(j)$	Γ_j
0		0.100000E 01	
1	0.25000000	0.400000E 01	4.00000000
2	0.39062500	0.102400E 02	2.55999997
3	0.48345947	0.211807E 02	2.06842569
4	0.55016300	0.384989E 02	1.81764312
5	0.60075133	0.640846E 02	1.66458224
6	0.64060120	0.100038E 03	1.56103358
7	0.67289308	0.148669E 03	1.48612019
8	0.69964281	0.212492E 03	1.42930074
9	0.72219642	0.294231E 03	1.38466483
10	0.74149013	0.396810E 03	1.34863561
12	0.77281414	0.677214E 03	1.29397216
16	0.81674511	0.163937E 04	1.22437218
20	0.84619291	0.335724E 04	1.18176362
30	0.88994993	0.131131E 05	1.12365872
40	0.91416065	0.358800E 05	1.09389962
50	0.92957572	0.798348E 05	1.07575957
60	0.94026696	0.155104E 06	1.06352773
70	0.94812379	0.273756E 06	1.05471459
80	0.95414462	0.449797E 06	1.04805914
90	0.95890736	0.699163E 06	1.04285359
100	0.96277000	0.103972E 07	1.03866966
150	0.97465568	0.487068E 07	1.02600335
200	0.98077881	0.147685E 08	1.01959789
300	0.98703395	0.715478E 08	1.01313636
400	0.99021478	0.220922E 09	1.00988190
500	0.99214144	0.531582E 09	1.00792079
600	0.99343379	0.109135E 10	1.00660960
700	0.99436089	0.200716E 10	1.00567108
800	0.99505845	0.340507E 10	1.00496608
1000	0.99603836	0.824695E 10	1.00397739
1500	0.99735114	0.412899E 11	1.00265588
2000	0.99801031	0.129744E 12	1.00199364
3000	0.99867143	0.652886E 12	1.00133032
4000	0.99900275	0.205699E 13	1.00099823
5000	0.99920180	0.501227E 13	1.00079882
6000	0.99933460	0.103798E 14	1.00066583
7000	0.99942952	0.192115E 14	1.00057080
8000	0.99950073	0.327502E 14	1.00049950
10000	0.99960048	0.798737E 14	1.00039968
20000	0.99980012	0.127523E 16	1.00019990
30000	0.99986672	0.645096E 16	1.00013329
40000	0.99990003	0.203802E 17	1.00009997
50000	0.99992002	0.497446E 17	1.00007997
60000	0.99993335	0.103134E 18	1.00006665
70000	0.99994286	0.191045E 18	1.00005713
80000	0.99995001	0.325886E 18	1.00004999
100000	0.99996001	0.795518E 18	1.00003999

FUNCTIONS USEFUL IN THE ARBITRARY CASE, FOR SELECTED j.
THE FUNCTIONS ARE EXPLAINED IN THE TEXT

BELLCOMM, INC.

<div align="center">

APPENDIX A

PROPERTIES OF THE SEQUENCE S_n

</div>

In this appendix, certain properties of the sequence are developed. They are, in the order in which they will be discussed, the convergence of the sequence, a proof that the sequence is monotonic increasing, and a determination of the limit of the sequence.

The nth term of the sequence is defined by

$$S_n = \frac{1}{n}\left[\sum_{j=1}^{n} \frac{1}{j^{1/2}}\right]^2 \qquad \text{A.1}$$

but it is more convenient to deal with

$$R_n = \frac{1}{n^{1/2}} \sum_{j=1}^{n} \frac{1}{j^{1/2}} \qquad \text{A.2}$$

where clearly

$$S_n = R_n^{\,2} \qquad \text{A.3}$$

We can write R_n in the form

$$R_n = \sum_{j=1}^{n} \frac{1}{(nj)^{1/2}}$$

Consider the term with $j = n - 1$; since we can write

$$n = (n - 1)^{1+2\epsilon} , \quad \epsilon > 0$$

then

$$[n(n - 1)]^{1/2} = (n - 1)^{1+\epsilon} \qquad\qquad A.4$$

Similarly, we have

$$(nj)^{1/2} = j^{1+\epsilon} \cdot I_j \qquad j = 2, \cdots, n-1$$

$$I_j \geq 1$$

The nth element of the sequence may then be written

$$R_n = \frac{1}{n} + \frac{1}{n^{1/2}} + \sum_{j=2}^{n-1} \frac{1}{j^{1+\epsilon}I_j}$$

$$\leq \frac{1}{n} + \frac{1}{n^{1/2}} + \sum_{j=1}^{n} \frac{1}{j^{1+\epsilon}}$$

since $\dfrac{1}{j^{1+\epsilon}I_j} \leq \dfrac{1}{j^{1+\epsilon}}$ for $I_j \geq 1$

However, the series $\displaystyle\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j^{1+\epsilon}}$ is bounded from above for any

$\epsilon>0$; let this bound be U_{ϵ}. We have

$$\lim_{n\to\infty} R_n \leq \frac{1}{n} + \frac{1}{n^{1/2}} + U_{\epsilon}$$

and we conclude that the sequence R_n, and hence also S_n, is bounded from above as $n\to\infty$.

We next wish to establish the relationship

$$S_{n+1} > S_n \qquad\qquad\qquad \text{A.5}$$

As before, we will work with R_n and show that

$$R_{n+1} > R_n$$

from which equation A.5 immediately follows. From the definition of R_n (equation A.2) we can derive a recursion relation

$$R_{n+1} = \left(\frac{n}{n+1}\right)^{1/2} R_n + \frac{1}{n+1} \qquad\qquad \text{A.6}$$

If we consider R_n to be a variable, R_{n+1} is linearly related to R_n, and it is easily shown that

$$R_{n+1} > R_n \text{ for } R_n < 1 + \left(\frac{n}{n+1}\right)^{1/2}$$

Similarly, of course, it follows

$$R_{n+2} > R_{n+1} \text{ for } R_{n+1} < 1 + \left(\frac{n+1}{n+2}\right)^{1/2}$$

Now, suppose we have shown that $R_n < 1 + \left(\frac{n}{n+1}\right)^{1/2}$; then it follows from equation A.6

$$R_{n+1} < \frac{n}{n+1} + \frac{1}{n+1} + \left(\frac{n}{n+1}\right)^{1/2}$$

so that

$$R_{n+1} < 1 + \left(\frac{n+1}{n+2}\right)^{1/2} \qquad\qquad \text{A.7}$$

That is, if we can show that for some R_j

$$R_j < 1 + \left(\frac{j}{j+1}\right)^{1/2} ,$$

we can apply A.7 successively and show

$$R_k < 1 + \left(\frac{k}{k+1}\right)^{1/2}, \quad k \geq j$$

But

$$R_1 = \frac{1}{(1)^{1/2}} \cdot \frac{1}{(1)^{1/2}} = 1 < 1 + \frac{1}{\sqrt{2}}$$

and we have established

$$R_j < 1 + \left(\frac{j}{j+1}\right)^{1/2} \qquad\qquad A.8$$

for all j, and therefore that

$$R_{n+1} > R_n$$

and that equation A.5 is valid.

We also have established an upper bound on R_n (equation A.8) and in the limit

$$\lim_{n\to\infty} R_n \leq \lim_{n\to\infty} \left[1 + \left(\frac{n}{n+1}\right)^{1/2} \right] = 2$$

So

$$\underset{n\to\infty}{\text{Lim}}\ S_n \leq 4 \qquad\qquad\qquad A.9$$

The limit of the sequence S_n can be established directly by means of a transformation as follows.*

Define y through the relation

$$y = \frac{j}{n}\ ;$$

$$\Delta y = \frac{j - (j - 1)}{n} = \frac{1}{n}$$

We note that the region $o \leq y \leq 1$ is divided by Δy into n equal lengths, and we can write

$$R_n = \sum_{y=o}^{y=1} \frac{\Delta y}{y^{1/2}}$$

In the limit as $n\to\infty$, the sum becomes an integral and

$$\underset{n\to\infty}{\text{Lim}}\ R_n = \int_0^1 \frac{dy}{y^{1/2}} = 2$$

---

*This method was suggested by B. G. Smith.

and we also have

$$\lim_{n \to \infty} S_n = 4$$

This is consistent with equation A.9.

BELLCOMM, INC.

APPENDIX B

In this appendix we prove the relationship

$$\Gamma_{j+1} < \Gamma_j \qquad\qquad B.1$$

The above relationship was used in Section III to prove that

$$P_{j+1} > P_j \qquad\qquad B.2$$

which indicates that it is advantageous to increase the number of transmission levels.

Γ_{j+1} is related to Γ_j through

$$\Gamma_{j+1} = \left(\frac{2\Gamma_j}{1 + \Gamma_j}\right)^2 \qquad\qquad B.3$$

Let us consider the function

$$\gamma_{j+1} = \left(\frac{2\gamma_j}{1 + \gamma_j}\right)^2 \qquad\qquad B.4$$

which generates Γ_{j+1} when $\gamma_j = \Gamma_j$; $\gamma_{j+1} = 1$ for $\gamma_j = 1$.

Appendix B

We may write

$$\Gamma_j = 1 + \int_1^{\Gamma_j} 1 \cdot d\gamma_j$$

B.5

$$\Gamma_{j+1} = 1 + \int_1^{\Gamma_j} \frac{d\gamma_{j+1}}{d\gamma_j} d\gamma_j$$

From B.4 we obtain

$$\frac{d\gamma_{j+1}}{d\gamma_j} = \frac{8\gamma_j}{(1 + \gamma_j)^3}$$

B.6

This function is plotted in Figure B-1. The properties which are important to the discussion are

$$\frac{d\gamma_{j+1}}{d\gamma_j} > 0 \qquad \text{for all finite } \gamma_j \geq 1 \qquad\qquad \text{B.7a}$$

$$\frac{d\gamma_{j+1}}{d\gamma_j} < 1 \qquad \text{for } \gamma_j > 1 \qquad\qquad \text{B.7b}$$

$$\frac{d\gamma_{j+1}}{d\gamma_j} = 1 \qquad \text{for } \gamma_j = 1 \qquad\qquad \text{B.7c}$$

Appendix B

Let us suppose for the moment that $\Gamma_j > 1$; by applying (B.7a) to the second of equations (B.5) we obtain

$$\Gamma_{j+1} > 1 \qquad\qquad \text{B.8}$$

Since $\Gamma_1 = 4$ (equation 26, Section III), the above relationship may be applied successively to Γ_2, Γ_3, $\cdot$ $\cdot$ $\cdot$, so that equation B.8 is valid for all j.

Compare the two equations B.5. Except for the point $\gamma_j = 1$, the integrand for Γ_{j+1} is less than that for Γ_j (equation B.7b and B.7c); since $\Gamma_j > 1$, however, we immediately conclude

$$\Gamma_{j+1} < \Gamma_j \qquad\qquad \text{B.9}$$
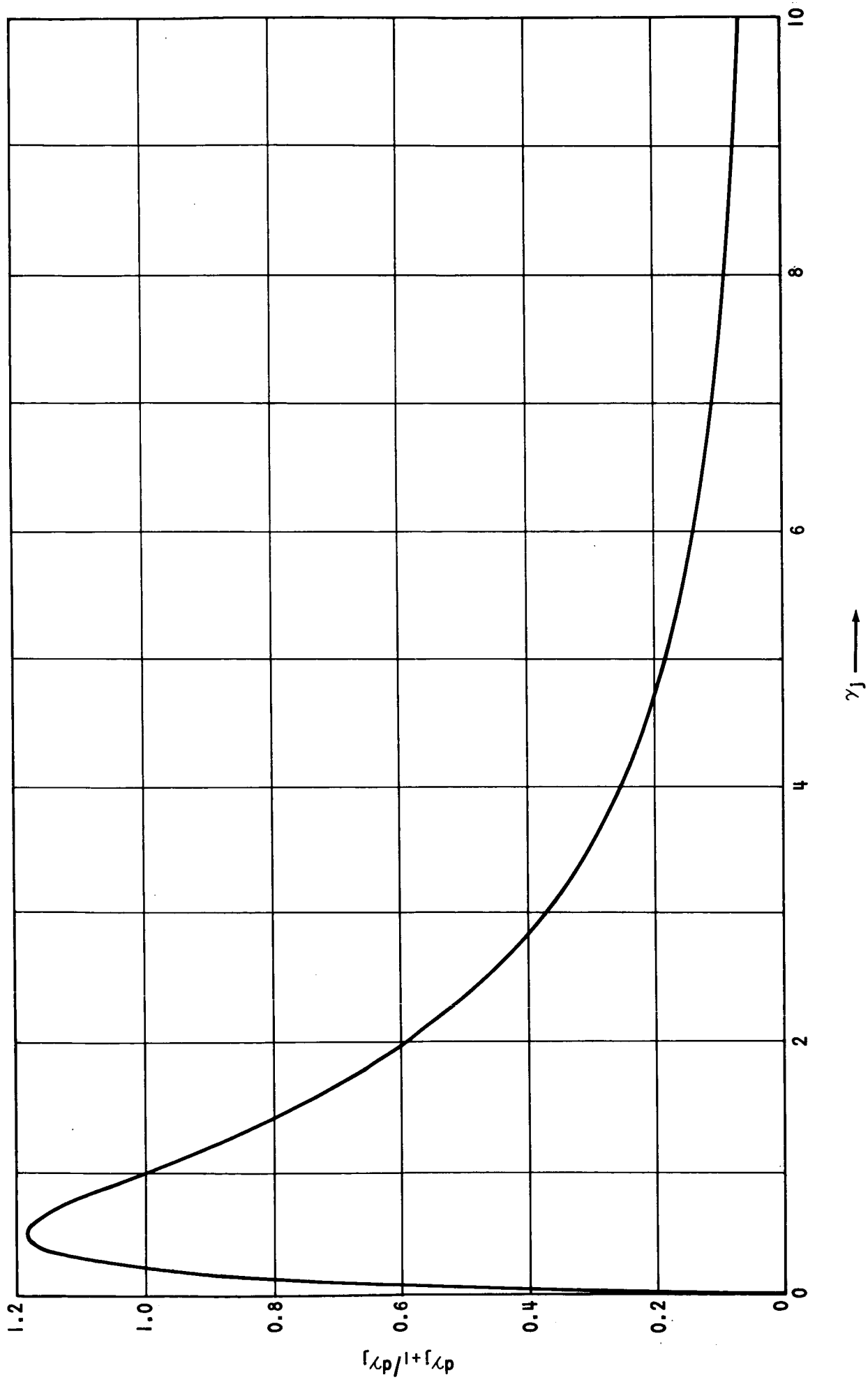
FIGURE B-1 - THE VALUE OF $d\gamma_{j+1}/d\gamma_j$ AS A FUNCTION OF γ_j (EQUATION B.6)